University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for five $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (20 points)
In the northern hemisphere at colatitude λ (as measured from the north pole, equivalent to north latitude $\frac{\pi}{2} - \lambda$), an ice rink is built by pouring water into an enclosure and then allowing it to freeze. If the rink is built this way, an isolated hockey puck that lies at rest on the ice won't move at all, even if the ice is frictionless (which is the case here).

The puck (of mass m) is tied to a frictionless center swivel using a taut massless rope of length R. The puck is set into counterclockwise uniform circular motion about the swivel point. As seen by an observer standing on the ice, the puck has a constant angular velocity ω about the swivel, *i.e.* it retraces its path around the ice every $\frac{2\pi}{\omega}$ seconds. The puck moves *slowly*: you may *not* assume that ω is much larger than Ω, the angular frequency ($= 2\pi/\text{day}$) of the earth's rotation about its axis.

What is the tension τ in the rope? You may work this problem either in the rest frame of the observer, or in the rest frame of the puck – but you must *state which frame you are using*.

Solution:
In the observer's frame, the puck is accelerating inward toward the swivel with acceleration $R\omega^2$, and it feels an outward Coriolis force equal to $2mR\Omega\cos\lambda$. Therefore

$$\tau = mR\big(\omega^2 + 2\omega\Omega\cos\lambda\big) \ .$$

In the puck's (*) frame, the vertical component of its angular velocity is $\omega^* = \omega + \Omega\cos\lambda$. This would seem to require a centrifugal force

$$\begin{aligned}\tau &= mR\omega^{*2} \\ &= mR\big(\omega^2 + 2\omega\Omega\cos\lambda + \Omega^2\cos^2\lambda\big) \ .\end{aligned}$$

However, the last term must be compensated by the normal force of the ice, since the tension must vanish if the puck isn't moving with respect to the ice ($\omega = 0$). Therefore

$$\tau = mR\big(\omega^2 + 2\omega\Omega\cos\lambda\big) \ .$$

2. (30 points)
A fixed upright solid cone with a height h and a circular base of radius R has a frictionless surface. The cone intercepts a vertical rain of tiny hailstones, which scatter elastically off the curved part of the cone. Since they are so tiny, only a negligible fraction of the hailstones hit the very tip of the cone. Neglect gravity.
(**a**) (10 points)
Show that the scattering angle Θ of the hailstones is 2α, where $\alpha = \arctan(R/h)$ is the half-angle of the cone. You may use this result in the remainder of the problem.
Solution:
For scattering off the cone, the hailstone's angle of incidence θ_{inc} must equal its angle of reflection θ_{refl}, because the frictionless surface of the cone can exert only a normal impulse on the hailstone. Since the angle of incidence is

$$\theta_{\text{inc}} = \frac{\pi}{2} - \alpha \ ,$$

the scattering angle is

$$\begin{aligned}\Theta &= \pi - \theta_{\rm inc} - \theta_{\rm refl} \\ &= \pi - 2(\tfrac{\pi}{2} - \alpha) \\ &= 2\alpha\ .\end{aligned}$$

(**b**) (10 points)
Using a purely geometrical argument, write down the total cross section σ_T for elastic scattering of a hailstone by the cone.
Solution:
As seen by the hailstones, the cone has a cross-sectional area equal to πR^2. Therefore

$$\sigma_T = \pi R^2\ .$$

(**c**) (10 points)
Taking ϕ to be the hailstone's angle about the cone's azimuth, write down the differential cross section

$$\frac{d^2\sigma}{\sin\Theta\, d\Theta\, d\phi}$$

for elastic scattering of a hailstone by the cone. [*Hint*: integrating the differential cross section over the full solid angle should yield σ_T.]
Solution:
From (**a**), the hailstones have $\Theta = 2\alpha$. Therefore the differential cross section is proportional to $\delta(\Theta - 2\alpha)$. Take the (unknown) constant of proportionality to be equal to C. Using the hint,

$$\begin{aligned}\sigma_T &= \int_0^\pi \sin\Theta\, d\Theta \int_0^{2\pi} d\phi\, \frac{d^2\sigma}{\sin\Theta\, d\Theta\, d\phi} \\ \pi R^2 &= \int_0^\pi \sin\Theta\, d\Theta \int_0^{2\pi} d\phi\, C\delta(\Theta - 2\alpha) \\ &= \int_0^\pi \sin\Theta\, d\Theta\, 2\pi\, C\delta(\Theta - 2\alpha) \\ &= 2\pi C \sin 2\alpha \\ C &= \frac{\pi R^2}{2\pi \sin 2\alpha} \\ \frac{d^2\sigma}{\sin\Theta\, d\Theta\, d\phi} &= \frac{R^2}{2\sin 2\alpha}\delta(\Theta - 2\alpha)\ .\end{aligned}$$

3. (40 points)
A physical system has a Lagrangian that is normalized (scaled) to be dimensionless. It is equal to

$$\mathcal{L}(a, b, \dot{a}, \dot{b}, t) = \tfrac{1}{2}\dot{a}^2 + \tfrac{1}{2}(a\dot{b})^2 - a^n\ ,$$

where a and b are dimensionless generalized coordinates, $a > 0$, n is an unspecified integer, and the time t is normalized so that it is dimensionless as well.
(**a**) (10 points)
Use one Euler-Lagrange equation to find the conserved canonical momentum p_0 in terms of a, b, $\dot{a}$, and $\dot{b}$.
Solution:

$$\begin{aligned}0 &= \frac{\partial\mathcal{L}}{\partial b} \\ \text{constant} &= \frac{\partial\mathcal{L}}{\partial\dot{b}} \\ p_0 &= a^2\dot{b}\ .\end{aligned}$$

(**b**) (10 points)
Write the other Euler-Lagrange equation. Substitute p_0 so that this equation is expressed entirely in terms of one of the generalized coordinates, its time derivatives, and constants.
Solution:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial a} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{a}} \\ -na^{n-1} + a\dot{b}^2 &= \ddot{a} \\ -na^{n-1} + a\left(\frac{p_0}{a^2}\right)^2 &= \ddot{a} \\ -na^{n-1} + \frac{p_0^2}{a^3} &= \ddot{a}\ .\end{aligned}$$

(**c**) (10 points)
Find a condition on n such that it is possible for the surviving generalized coordinate in part (**b**) to be constant.
Solution:

$$\begin{aligned}-na^{n-1} + \frac{p_0^2}{a_0^3} &= 0 \\ \frac{p_0^2}{a_0^3} &= na_0^{n-1} \\ n &> 0\ .\end{aligned}$$

(**d**) (10 points)

If the "constant" generalized coordinate in part **(c)** were perturbed slightly, would it oscillate stably about its "constant" value? Explain.
Solution:
Examining the structure of the Lagrangian, the actual (normalized) potential is a^n and the pseudopotential is $\frac{1}{2}(a\dot{b})^2 = \frac{1}{2}p_0^2/a^2$. The effective potential, which is the sum of the two, is equal to $+\infty$ both at $a = 0$ and at $a = \infty$. Thus the extremum in the effective potential, where it is possible for a to be constant, is a minimum rather than a maximum. Therefore a is stable about its constant value.

Alternatively, the effective potential may be differentiated twice to get the effective spring constant, or the method of perturbations may be applied.

4. (45 points)
A double pendulum consists of a top bob of mass $3m$, hung from the ceiling by a string of length ℓ; and a bottom bob of mass m, hung from the top bob by another string of length ℓ. The top string makes an angle $\phi \ll 1$ from the vertical direction; the bottom string makes an angle $\theta \ll 1$, also measured from the vertical direction. Note that the two bobs have different masses.
(**a**) (10 points)
If the kinetic energy is expressed in units of $m\ell^2$, the potential energy is expressed in units of $mg\ell$, and the time is expressed in units of $\sqrt{\ell/g}$, making all possible small-angle approximations, show that (within an additive constant) the Lagrangian can be written

$$\mathcal{L} = \tfrac{1}{2}\left(4\dot{\phi}^2 + 2\dot{\phi}\dot{\theta} + \dot{\theta}^2 - 4\phi^2 - \theta^2\right) .$$

You may use this result for the remainder of the problem.
Solution:
Let (x, y) be the coordinates of the top bob, and (u, v) be the coordinates of the bottom bob, with respect to their respective equilibrium points. Then

$$\begin{aligned} x &= \ell \sin\phi \approx \ell\phi \\ y &= \ell(1 - \cos\phi) \approx \tfrac{1}{2}\ell\phi^2 \\ u &= x + \ell\sin\theta \approx x + \ell\theta \\ v &= u + \ell(1 - \cos\theta) \approx y + \tfrac{1}{2}\ell\theta^2 . \end{aligned}$$

The kinetic energy is

$$\begin{aligned} \frac{2T}{m\ell^2} &= 3(\dot{x}^2 + \dot{y}^2) + \dot{u}^2 + \dot{v}^2 \\ &\approx 3\dot{x}^2 + \dot{u}^2 \\ &= 3\dot{\phi}^2 + (\dot{\phi} + \dot{\theta})^2 \\ &= 4\dot{\phi}^2 + 2\dot{\phi}\dot{\theta} + \dot{\theta}^2 . \end{aligned}$$

The potential energy is

$$\begin{aligned} \frac{2U}{mg\ell} &= 3y + v \\ &= 3\phi^2 + (\phi^2 + \theta^2) \\ &= 4\phi^2 + \theta^2 . \end{aligned}$$

Therefore the normalized Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\left(4\dot{\phi}^2 + 2\dot{\phi}\dot{\theta} + \dot{\theta}^2 - 4\phi^2 - \theta^2\right) .$$

(**b**) (15 points)
Expressed as a ratio to $\sqrt{g/\ell}$, find the angular frequencies of this system's normal modes. [*Hint*: the winding number of this system turns out to be $\sqrt{3}$.]
Solution:
Applying the Euler-Lagrange equations yields

$$\begin{aligned} -4\phi &= 4\ddot{\phi} + \ddot{\theta} \\ -\theta &= \ddot{\theta} + \ddot{\phi} . \end{aligned}$$

Looking for solutions of the form

$$\begin{aligned} \phi &= \phi_0 \cos\omega t \\ \theta &= \phi_0 \cos\omega t , \end{aligned}$$

we obtain

$$\begin{aligned} 0 &= 4(1 - \omega^2)\phi_0 - \omega^2\theta_0 \\ 0 &= (1 - \omega^2)\theta_0 - \omega^2\phi_0 . \end{aligned}$$

The secular determinant must vanish:

$$\begin{aligned} 0 &= 4(1 - \omega^2)^2 - \omega^2 \\ &= 3\omega^4 - 8\omega^2 + 4 \\ \omega^2 &= \frac{3 \pm 2}{6} \\ \omega &= \sqrt{\frac{2}{3}} \text{ or } \sqrt{2} . \end{aligned}$$

(**c**) (10 points)
If you were unable to make the approximations $\phi \ll 1$ and $\theta \ll 1$, you would need to solve this system numerically. This can be easier if you use a set of first-order coupled partial differential equations, rather than a set of second-order partial differential equations. Given the Lagrangian, how would you obtain this first-order set of equations? (Only an explanation of what you would do is required.)
Solution:
The desired set of first-order coupled partial differential equations are Hamilton's equations

$$\begin{aligned} \dot{\phi} &= \frac{\partial \mathcal{H}}{\partial p_\phi} \\ \dot{p}_\phi &= -\frac{\partial \mathcal{H}}{\partial \phi} \\ \dot{\theta} &= \frac{\partial \mathcal{H}}{\partial p_\theta} \\ \dot{p}_\theta &= -\frac{\partial \mathcal{H}}{\partial \theta} \ . \end{aligned}$$

The canonical momenta are

$$\begin{aligned} p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \ . \end{aligned}$$

The Hamiltonian is

$$\mathcal{H} = \dot{\phi} p_\phi + \dot{\theta} p_\theta - \mathcal{L} \ ,$$

where $\mathcal{H}$ must be reexpressed in terms of ϕ, θ, p_ϕ, and p_θ.
(**d**) (10 points)
Again for the conditions of part (**c**) (ϕ and θ not necessarily small), assume that you have an ideal double pendulum, an arbitrarily fast and precise computer, and "fairly accurate" initial conditions. Would you expect to obtain a "fairly accurate" prediction for its motion? Would your expecations depend on the range of motion that is considered? Explain.
Solution:
When the double pendulum with equal bob masses was simulated numerically by Hand, he found that the difference between solutions for infinitesimally different initial conditions grew exponentially with the number of periods considered, when the initial angle of the top bob was large (90°). Both Hand's pendulum and the present pendulum have irrational winding numbers, so we assume that the behavior of the current pendulum would be similar: when the initial angles are sufficiently large, "fairly accurate" predictions can't be made, even if the initial conditions are known with fair accuracy.

5. (40 points)
A physical system is described by a single dimensionless generalized coordinate $b(s,t)$ that is a function of two independent variables: a time variable t and a (one-dimensional) field variable s. When s and t are normalized (scaled) to be dimensionless, and the Lagrangian density $\mathcal{L}'$ is similarly renormalized, $\mathcal{L}'$ takes the form

$$\mathcal{L}'(b, \tfrac{\partial b}{\partial s}, \tfrac{\partial b}{\partial t}, s, t) = \tfrac{1}{2}\left(\tfrac{\partial b}{\partial s}\right)^2 - \tfrac{1}{2}\left(\tfrac{\partial b}{\partial t}\right)^2 \ .$$

(**a**) (10 points)
Using the version of the Euler-Lagrange equation that is appropriate for a Lagrangian density, show that the equation controlling the evolution of $b(s,t)$ is

$$\tfrac{\partial^2 b}{\partial s^2} - \tfrac{\partial^2 b}{\partial t^2} = 0 \ .$$

You may use this result in the remainder of this problem.
Solution:

$$\frac{d}{ds}\frac{\partial \mathcal{L}'}{\partial \frac{\partial b}{\partial s}} + \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \frac{\partial b}{\partial t}} = \frac{\partial \mathcal{L}'}{\partial b}$$

$$\tfrac{\partial^2 b}{\partial s^2} - \tfrac{\partial^2 b}{\partial t^2} = 0 \ .$$

This is a wave equation with unit phase velocity.
(**b**) (10 points)
If $-\infty < s < \infty$, *i.e.* there are no boundaries for s, what is the *general* solution $b(s,t)$ to this equation?
Solution:

$$b(s,t) = b_+(s-t) + b_-(s+t) \ ,$$

where b_+ and b_- are any two differentiable functions of their arguments. This describes a shape

b_+ propagating in the $+s$ direction, and a shape b_- propagating in the $-s$ direction, each with phase velocity equal (in these coordinates) to unity.

(**c**) (10 points)
Now impose the boundary condition

$$b(s=0,t) = b(s=1,t) = 0\ .$$

What are the angular frequencies of the normal modes of this system?

Solution:
When boundary conditions are imposed, the solutions become sinusoidal standing waves of the form

$$b(s,t) \propto \sin ks \cos kt\ ,$$

where the harmonic functions are chosen to satisfy the particular boundary conditions that are imposed. Here we choose $\sin ks$ because $b(s=0)=0$. The boundary condition $b(s=1)=0$ is satisfied for $k=\pi,\ 2\pi,\ 3\pi...$ Therefore the normal angular frequencies are $\omega=\pi,\ 2\pi,\ 3\pi...$

(**d**) (10 points)
Finally, retaining the boundary condition introduced in part (**c**), impose the initial conditions

$$\begin{aligned} b(s,t=0) &= \sin \pi s \\ \tfrac{\partial b}{\partial t}(s,t=0) &= 0\ . \end{aligned}$$

What is the earliest time t_0 such that

$$b(s,t_0) = -b(s,t=0)\ ,$$

i.e. the field $b(s,t)$ reverses sign but is otherwise unchanged? Explain your reasoning.

Solution:
The boundary condition is such that only the first Fourier component k_1 is excited. At $k_1 t_0 = \pi$, it will change sign. Therefore, since $k_1=\pi$, $t_0=1$.

Alternatively, you may use the fact that half of the initial waveform propagates to the left and half to the right. Each is inverted at the boundary. After a total propagation distance of $\frac{1}{2}+\frac{1}{2}=1$, the two inverted waveforms recombine. Again the elapsed time is $t_0=1$.

6. (25 points)
A one-dimensional physical system with generalized coordinate q and canonically conjugate momentum p is described by a Hamiltonian $\mathcal{H}(q,p,t)$ that is a smooth function of the variables upon which it depends. This is a conservative system (no dissipation), so that $d\mathcal{H}/dt$ vanishes, *i.e.* $\mathcal{H}$ is a constant of the motion.

(**a**) (10 points)
Prove that $\partial\mathcal{H}/\partial t$ vanishes.

Solution:

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \frac{\partial\mathcal{H}}{\partial t} + \frac{\partial\mathcal{H}}{\partial q}\dot{q} + \frac{\partial\mathcal{H}}{\partial p}\dot{p} \\ 0 &= \frac{\partial\mathcal{H}}{\partial t} - \dot{p}\dot{q} + \dot{q}\dot{p} \\ 0 &= \frac{\partial\mathcal{H}}{\partial t}\ , \end{aligned}$$

where Hamilton's equations are used in the next to last line.

(**b**) (15 points)
This system also is characterized by a different smooth function $F(q,p,t)$ of the same variables. It is known that F is also a constant of the motion. The quantity $\partial F/\partial t$ describes the *explicit* time dependence of the function F; it can be nonzero even when F is conserved. Prove that $\partial F/\partial t$ is a constant of the motion.

Solution:

$$\begin{aligned} 0 = \frac{dF}{dt} &= \frac{\partial F}{\partial t} + [F,\mathcal{H}] \\ \frac{d}{dt}\frac{\partial F}{\partial t} &= \frac{d}{dt}[\mathcal{H},F] \\ \frac{d}{dt}\frac{\partial F}{\partial t} &= [\dot{\mathcal{H}},F] + [\mathcal{H},\dot{F}] \\ &= [0,F] + [\mathcal{H},0] \\ &= 0\ . \end{aligned}$$